Chapter 1 - Notation and Review of Newton's Laws

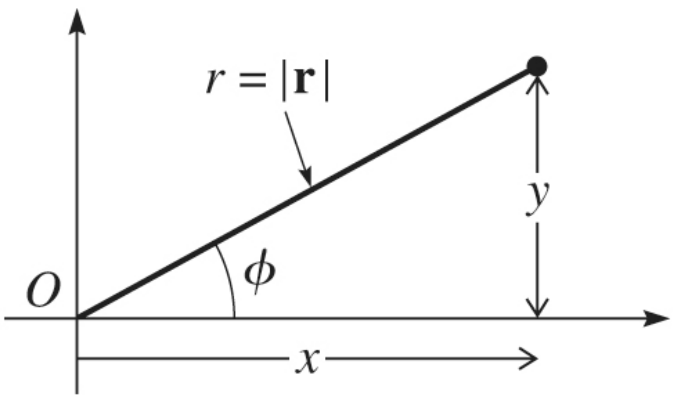
E. 2D polar coordinates
  - Polar unit vectors and their derivatives
  - Newton's 2nd Law in polar coordinates

F. 3D Cylindrical coordinates
  - Cylindrical unit vectors and their derivatives
  - Newton's 2nd Law in cylindrical coordinates

E. 2D Polar Coordinates

2D Polar Coordinates

Polar coordinates (r, ϕ) use the radial distance r from the origin and the angle ϕ from the positive x axis and coordinates.



Transformation equations between Cartesian coordinates (x, y) and Polar Coordinates (r, ϕ):

$(r, \phi) \longrightarrow (x, y)$

$$x = r\cos\phi$$
$$y = r\sin\phi$$

$(x, y) \longrightarrow (r, \phi)$

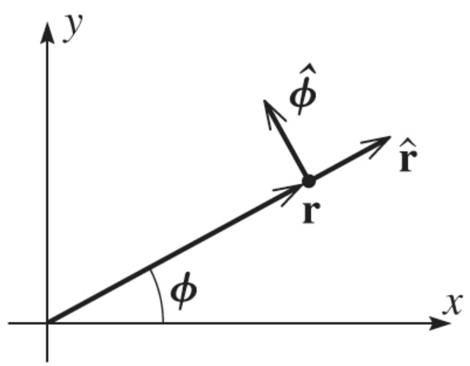
$$r = \sqrt{x^2 + y^2}$$
$$\phi = \arctan(y/x)$$

2D Polar Coordinates

As in Cartesian coordinates, polar unit vectors point in the direction that the corresponding coordinate increases.

- the r unit vector $\hat{r}$ points radially outward from the origin
- the ϕ unit vector $\hat{\phi}$ points counter-clockwise in the direction of increasing ϕ

Polar Unit Vectors

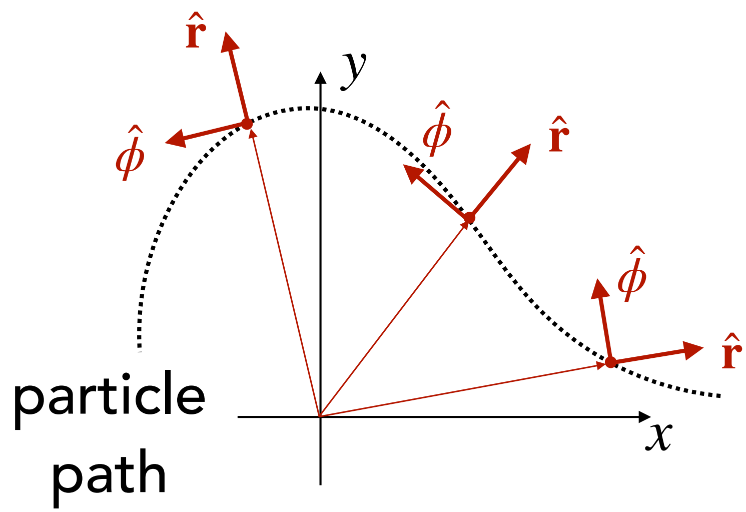


$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{|\mathbf{r}|} \quad \rightarrow \quad \mathbf{r} = r\hat{\mathbf{r}}$$

$$\hat{\phi} = \hat{\mathbf{z}} \times \hat{\mathbf{r}}$$ ($\hat{\mathbf{z}}$ points out of x-y plane)

2D Polar Coordinates

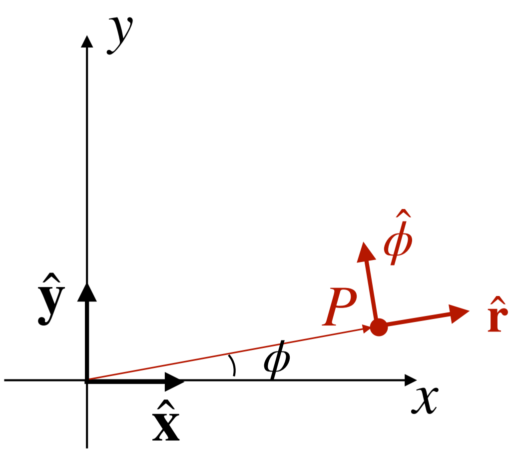
Unlike Cartesian unit vectors, the Direction of $\hat{\mathbf{r}}$ and $\hat{\phi}$ change as one moves in the (x,y) plane.



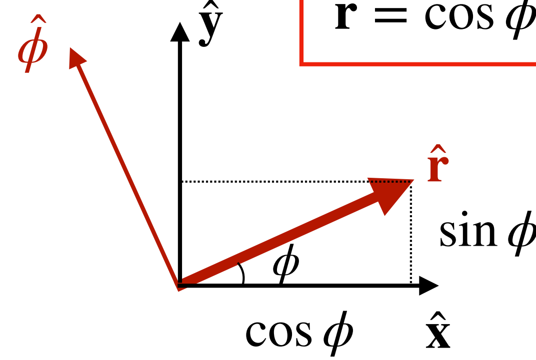
Imagine following the dotted path. The (r, ϕ) unit vectors continuously change direction as time passes.

2D Polar Coordinates

We can use trig to write down expressions for $\hat{\mathbf{r}}$ and $\hat{\phi}$ in terms of $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$. Consider point P in the diagram below:

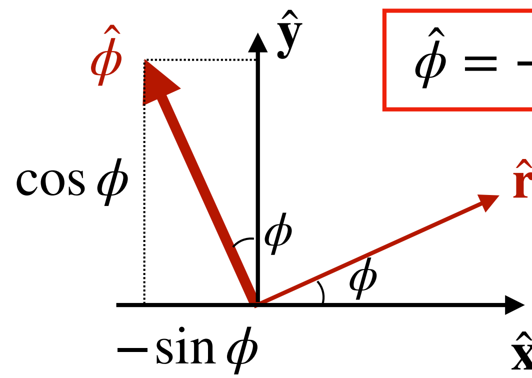


Solve for $\hat{\mathbf{r}}$:



$$\hat{\mathbf{r}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}$$

Solve for $\hat{\phi}$:



$$\hat{\phi} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}$$

2D Polar Coordinates

Goal: Calculate the velocity v of an object in polar coordinates.

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{d}{dt}(r\hat{\mathbf{r}}) \qquad \text{(we substituted} \quad \mathbf{r} = r\hat{\mathbf{r}}\ \text{)}$$

Because the polar unit vectors rotate as a given point moves in the plane, we cannot treat $\hat{\mathbf{r}}$ as a constant in this equation. Thus, using the product rule, we have

$$\dot{\mathbf{r}} = r\dot{\hat{\mathbf{r}}} + \dot{r}\hat{\mathbf{r}}$$

In order to proceed we need to calculate the time derivative of each unit vector:

$$\dot{\hat{\mathbf{r}}} = \frac{d\hat{\mathbf{r}}}{dt} = ? \qquad \dot{\hat{\phi}} = \frac{d\hat{\phi}}{dt} = ?$$

2D Polar Coordinates

Time derivative of $\hat{\mathbf{r}}$:

Start with $\hat{\mathbf{r}} = \cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}}$

Differentiate: $$\dot{\hat{\mathbf{r}}} = \frac{d\hat{\mathbf{r}}}{dt} = \frac{d}{dt}\left[\cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}}\right]$$

$$= \left[-\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}\right]\dot{\phi}$$

$$= \dot{\phi}\hat{\phi}$$

We see this expression is what we derived for $\hat{\phi}$ two slides ago: $\hat{\phi} = -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}$

Thus, $$\dot{\hat{\mathbf{r}}} = \dot{\phi}\hat{\phi}$$

Similarly, the time derivative of $\hat{\phi}$ is

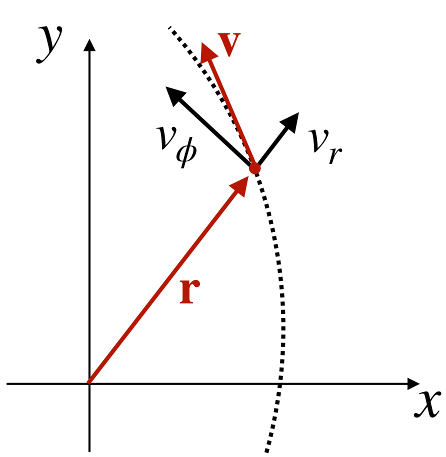
$$\dot{\hat{\phi}} = -\dot{\phi}\hat{\mathbf{r}}$$

2D Polar Coordinates

We can now evaluate the **velocity vector** in polar coordinates

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\hat{\mathbf{r}}}$$

$$= \dot{r}\hat{\mathbf{r}} + r(\dot{\phi}\hat{\phi}) \qquad \text{(use} \quad \dot{\hat{\mathbf{r}}} = \dot{\phi}\hat{\phi}\text{)}$$



$$\boxed{\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\phi}}$$

The velocity has two orthogonal components:

$v_r = \dot{r}$ radial velocity component

$v_\phi = r\dot{\phi} = r\omega$ tangential velocity component, where $\omega = \dot{\phi}$ is the angular velocity

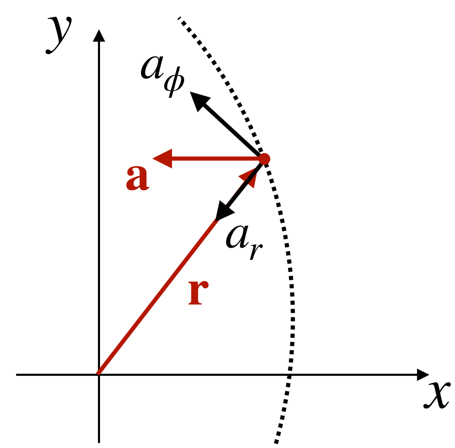
2D Polar Coordinates

The acceleration vector in polar coordinates is found similarly:

$$\mathbf{a} = \dot{\mathbf{v}} = \frac{d}{dt}\left[\dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\phi}\right]$$

$$= \left(\ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\hat{\mathbf{r}}}\right) + \left(\left(\dot{r}\dot{\phi} + r\ddot{\phi}\right)\hat{\phi} + r\dot{\phi}\dot{\hat{\phi}}\right)$$

$$\boxed{\mathbf{a} = \left(\ddot{r} - r\dot{\phi}^2\right)\hat{\mathbf{r}} + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\hat{\phi}}$$



The acceleration has two orthogonal components:

$a_r = \ddot{r} - r\dot{\phi}^2$ radial acceleration (note: $r\dot{\phi}^2$ is the centripetal acc.)

$a_\phi = r\ddot{\phi} + 2\dot{r}\dot{\phi}$ azimuthal (or tangential) acceleration

Newton's second law in polar coordinates

Newton's second law:

$$\mathbf{F} = m\mathbf{a}$$

Write force and acceleration vectors in terms of polar components:

$$\mathbf{F} = F_r\hat{\mathbf{r}} + F_\phi\hat{\phi} \qquad \mathbf{a} = a_r\hat{\mathbf{r}} + a_\phi\hat{\phi}$$

where

$$a_r = \ddot{r} - r\dot{\phi}^2$$
$$a_\phi = r\ddot{\phi} + 2\dot{r}\dot{\phi}$$

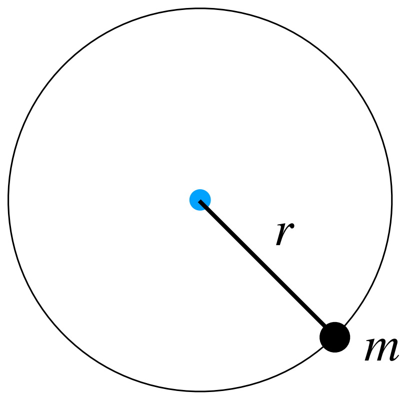
Equate radial and azimuthal components:

radial (r): $F_r = ma_r \;\longrightarrow\; F_r = m\left(\ddot{r} - r\dot{\phi}^2\right)$

azimuthal (ϕ): $F_\phi = ma_\phi \;\longrightarrow\; F_\phi = m\left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)$

Example: Motion on a Circle

Mass m is constrained to move on a circle with fixed radius r.

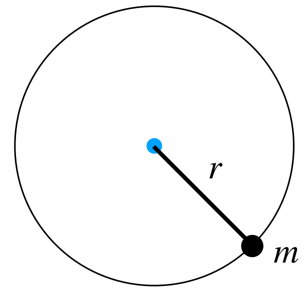


a) Write down Newton's 2nd law in polar coordinates given this constraint.
b) Assume the angular position as a function of time is $\phi(t) = at^3$, where a has units of rad/s^3. Solve for the radial and azimuthal force components acting on the particle to produce this motion.

Try it on your own. The solution is on the next 2 slides.

Example: Motion on a Circle

a) Write down Newton's 2nd law in polar coordinates assuming the motion is constrained to move in a circle with radius r.



Solution.

Newton's 2nd law in polar coordinates:

$$F_r = m\left(\ddot{r} - r\dot{\phi}^2\right)$$
$$F_\phi = m\left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)$$

Because the particle can not move in the radial direction, both $\dot{r} = 0$ and $\ddot{r} = 0$. We can remove those terms from the F=ma equation:

$$F_r = m\left(\cancel{\ddot{r}} - r\dot{\phi}^2\right)$$
$$F_\phi = m\left(r\ddot{\phi} + 2\cancel{\dot{r}}\dot{\phi}\right)$$

$$\boxed{\begin{aligned} F_r &= -mr\dot{\phi}^2 \\ F_\phi &= mr\ddot{\phi} \end{aligned}}$$

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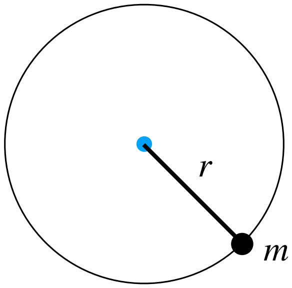
Centripetal force keeping particle on circular path

Azimuthal force driving rotational acceleration

Example: Motion on a Circle

b) the angular position as a function of time is $\phi(t) = at^3$, where a has units of rad/s^3. Solve for the radial and azimuthal force components acting on the particle to produce this motion.



Solution.

We apply the result from the previous slide:

$$F_r = -mr\dot{\phi}^2$$
$$F_\phi = mr\ddot{\phi}$$

We evaluate the derivatives:

$$\dot{\phi} = \frac{d}{dt}\left(at^3\right) = 3at^2$$

$$\ddot{\phi} = \frac{d^2}{dt^2}\left(at^3\right) = 6at$$

Plug into the F=ma equations above:

$$F_r = -mr(3at^2)^2 = -9ma^2rt^4$$

$$F_\phi = mr(6at) = 6mart$$

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F. 3D Cylindrical Coordinates

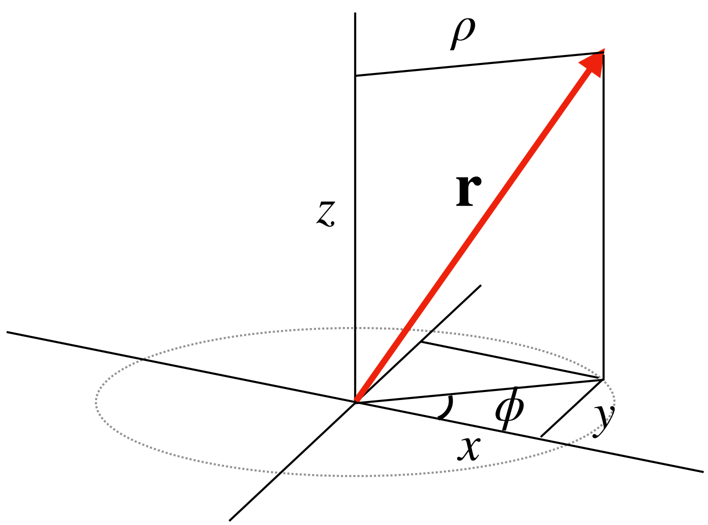
3D Cylindrical Coordinates

Cylindrical Coordinates:

z = height above the x-y plane

ρ = distance from the origin projected on the x-y plane

ϕ = angle measured from positive x axis



Transformation equations:

$$(\rho, \phi, z) \longrightarrow (x, y, z)$$

$$x = \rho \cos \phi$$

$$y = \rho \sin \phi$$

$$z = z$$

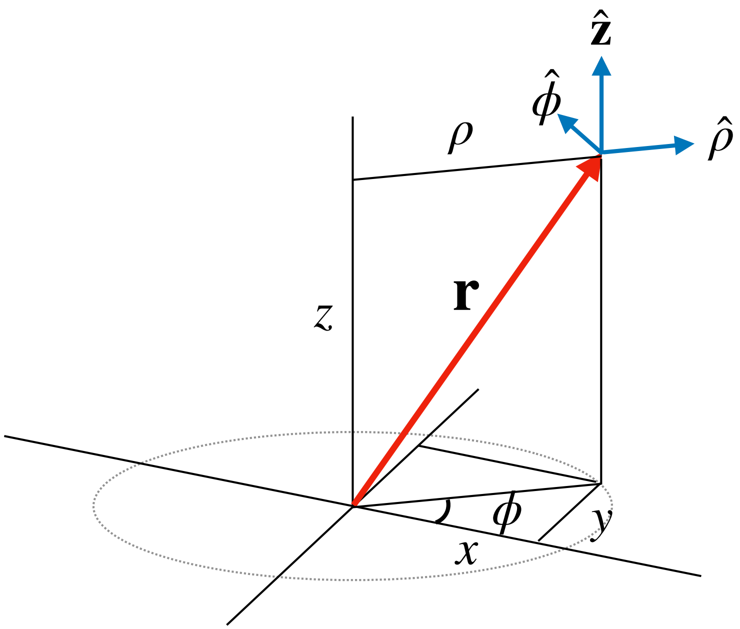
3D Cylindrical Coordinate Unit Vectors

Cylindrical Coordinate Unit Vectors:

$\hat{z}$ = points upward, perpendicular to the x-y plane

$\hat{\rho}$ = points outward from origin in the x-y plane

$\hat{\phi}$ = azimuthal direction in the x-y plane



Cylindrical Unit Vectors and their derivatives:

$$\hat{\rho} = \hat{x}\cos\phi + \hat{y}\sin\phi \qquad \dot{\hat{\rho}} = \dot{\phi}\hat{\phi}$$

$$\hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi \qquad \dot{\hat{\phi}} = -\dot{\phi}\hat{\rho}$$

$$\hat{z} = \hat{z} \qquad \dot{\hat{z}} = 0$$

Newton's Second Law in Cylindrical Coordinates

We follow method for deriving Newton's second law for polar coordinates. The result is:

radial (ρ):	$F_\rho = ma_\rho$	$\longrightarrow$	$F_\rho = m\left(\ddot{\rho} - \rho\dot{\phi}^2\right)$
azimuthal (ϕ):	$F_\phi = ma_\phi$	$\longrightarrow$	$F_\phi = m\left(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}\right)$
vertical (z):	$F_z = ma_z$	$\longrightarrow$	$F_z = m\ddot{z}$